Let $\{U_\alpha\}_{\alpha\in J}$ be an indexed open covering of X. An indexed family of cont. funs $\psi_\alpha : X \to [0,1]$ is said to be a partition of unity on X, dominated by $\{U_\alpha\}$, if

(1) $(\operatorname{Supp}\psi_\alpha) \subset U_\alpha$, $\forall \alpha\in J$

(2) $\{\operatorname{Supp}\psi_\alpha\}_{\alpha\in J}$ is locally finite

(3) $\sum_\alpha \psi_\alpha(x) = 1$, $\forall x\in X$ (we need (2) in order for (3) to make sense)

Goal: Let X be a paracpt Hausdorff space, let $\{U_\alpha\}_{\alpha\in J}$ be an indexed open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}$

Lemma 41.6 (Shrinking Lemma)

X: paracpt, Hausdorff. $\{U_\alpha\}_{\alpha\in J}$: indexed open covering. Then there exists a locally finite indexed family $\{V_\alpha\}_{\alpha\in J}$ of open sets covering X s.t. $\overline{V_\alpha} \subset U_\alpha$, $\forall \alpha\in J$.

Pf: Thm 41.1 says X is normal. Thus X is regular.

now we proceed similar to the proof in Lemma 41.3 (2)⇒(3):

Let $\mathcal{A}$ be the collection of all open sets A of X s.t. $\overline{A}$ is contained in some U_α. Regularity of X implies that $\mathcal{A}$ covers X. Since X is paracpt, we can find a locally finite open refinement $\mathcal{B}$ of $\mathcal{A}$ that covers X.

Let us index $\mathcal{B}$ with some index set I and denote element of $\mathcal{B}$ by B_β, $\beta\in I$. Thus $\{B_\beta\}_{\beta\in I}$ is a locally finite indexed family.

可能有好幾個，反正選一個

Define a map $f: I \to J$, $\beta \mapsto f(\beta)$ by $\overline{B_\beta} \subset U_{f(\beta)}$

then for each $\alpha \in J$, we define V_α to be the union of the elements of the collection $\mathcal{B}_\alpha = \{ B_\beta \mid f(\beta) = \alpha \}$

i.e. $V_\alpha := \bigsqcup_{f(\beta)=\alpha} B_\beta$,

lemma 39.1

def of f : $\overline{B_\beta} \subset U_{f(\beta)}$

then $\overline{V_\alpha} = \overline{\bigsqcup_{f(\beta)=\alpha} B_\beta} = \bigsqcup_{f(\beta)=\alpha} \overline{B_\beta} \subset U_\alpha$, $\forall \alpha \in J$

To show that $\{V_\alpha\}$ is locally finite:

Since $\mathcal{B}$ is locally finite, so $\forall x \in X$, $\exists$ nbd W_x s.t. W_x intersects finitely many elements from $\mathcal{B}$, say $B_1, \dots, B_k$.

Then W_x can intersects $V_\alpha = \bigsqcup_{f(\beta)=\alpha} B_\beta$ only if $\alpha = f(1)$, or $\dots$, or $f(k)$

$\therefore$ W_x can intersects only finitely many elements from $\{V_\alpha\}_{\alpha \in J}$

#

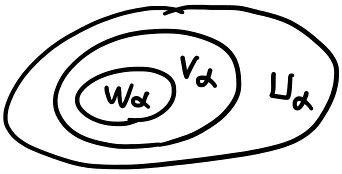
Now we should proceed to our goal:

Thm 41.7 X paracpt, Hausdorff, let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of X. Then there exists a partition of unity on X dominated by $\{U_\alpha\}$.

pf: By Shrinking Lemma, we have a locally finite indexed family of open sets $\{V_\alpha\}$ s.t. $\overline{V_\alpha} \subset U_\alpha$. Apply this again we have another family $\{W_\alpha\}$ s.t. $\overline{W_\alpha} \subset V_\alpha$.

paracpt + Hausdorff $\Rightarrow$ X is normal, so Urysohn Lemma:

$\exists$ cont $f: X \to [a,b]$ s.t. $f(x) = a$, $f(y) = b$, $\forall x \in A, y \in B$, A, B are disjoint closed subsets of X



so for each pair (W_α, V_α), $\alpha \in J$, $\exists f_\alpha$ cont, $X \longrightarrow [0,1]$
s.t. $f_\alpha|_{\overline{W_\alpha}} \equiv 1$, $f_\alpha|_{X \setminus V_\alpha} \equiv 0$ $\Rightarrow (\operatorname{supp} f_\alpha) \subset V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$

To show that $\{\operatorname{supp} f_\alpha\}_{\alpha \in J}$ is locally finite, first notice that $\{\overline{V_\alpha}\}$ is locally finite b/c $\{V_\alpha\}$ is ($\because$ if $U \cap \overline{V_\alpha} \neq \phi$ then $U \cap V_\alpha \neq \phi$) and since $\{\operatorname{supp} f_\alpha\} \subset \overline{V_\alpha}$, this means $\{\operatorname{supp} f_\alpha\}$ is also locally finite.

lastly, for the sum of f_α. First notice that, $\forall x \in X$, $\exists$ at least one f_α s.t. $f_\alpha(x) > 0$ ($\because \{W_\alpha\}$ covers X)
Define $g(x) = \sum_\alpha f_\alpha(x)$. Since $\{\operatorname{supp} f_\alpha\}$ is locally finite, it means $\forall x \in X$, $\exists$ nbd $\mathcal{O}_x$ s.t. $\mathcal{O}_x$ intersects only finitely many $\{\operatorname{supp} f_\alpha\}$ i.e. only finitely many f_α is nonzero on $\mathcal{O}_x$ $\Rightarrow g|_{\mathcal{O}_x}$ equals to a finite sum of cont. fun which is again cont. on $\mathcal{O}_x$ $\Rightarrow g$ is cont on X.

And $g(x) > 0$ $\forall x \in X$ $\Rightarrow$ Define $\Phi_\alpha(x) = \dfrac{f_\alpha(x)}{g_\alpha(x)}$

$\therefore$ of course $\sum_\alpha \Phi_\alpha(x) \equiv 1$.

\#

(note that $\{\operatorname{supp} \Phi_\alpha\} = \{\operatorname{supp} f_\alpha\}$)

Thm 34.2 (Imbedding thm)

Let X be a space in which one-pt sets are closed. Suppose that $\{f_\alpha\}_{\alpha\in J}$ is an indexed family of cont. funs $f_\alpha : X \to \mathbb{R}$ satisfying that, $\forall x_0 \in X$ and each nbd U of x_0, $\exists$ an index α such that $f_\alpha(x_0) > 0$ and $f_\alpha|_{X\setminus U} = 0$. Then the fun $F: X \to \mathbb{R}^J := \prod_{\alpha\in J} A_\alpha$, each $A_\alpha = \mathbb{R}$, with product top defined by $F(x) = (f_\alpha(x))_{\alpha\in J}$ is an imbedding of X in $\mathbb{R}^J$.

Pf: To prove that $F|_X : X \to F(X)$ is a homeomorphism, we need to show that F is cont, injective, and F^{-1} is cont.

① F is cont:

Recall Thm 19.6, F is cont iff f_α is cont. $\forall \alpha$.

($\prod_\alpha A_\alpha$ with product top.)

② F is injective:

if $x \neq y$, then since one-pt sets are closed, so $X\setminus\{y\}$ is open, and is a nbd of x, so by assumption of the thm, $\exists$ index β, s.t. $f_\beta(x) > 0$ and $f_\beta|_{X\setminus(X\setminus\{y\})} = 0$ i.e. $f_\beta(y) = 0$

i.e. $F(x) = (f_\alpha(x))_{\alpha\in J} \neq F(y) = (f_\alpha(y))_{\alpha\in J}$

③ F^{-1} is cont.

Recall the equivalent def for continuity: Denote $F(X)$ by Z.

Claim: $\forall z_0 \in Z$, and any nbd U containing $F^{-1}(z_0)$, there exists a nbd W of z_0, s.t. $F^{-1}(W) \subset U$.

$\forall z_0 \in Z := F(X)$, denote $F^{-1}(z_0)$ by x_0 (this is unique b/c F injective)

Let U be any nbd of x_0.

Then by the assumption of the thm, $\exists$ index γ s.t. $f_\gamma(x_0) > 0$ and $f_\gamma|_{X \setminus U} = 0$.

Recall projection maps, $\pi_\alpha : \prod_{\alpha \in J} A_\alpha \to A_\alpha$, to its α-th coordinate.

Consider the preimage $\pi_\gamma^{-1}((0, +\infty))$ i.e. $\pi_\gamma^{-1}(\mathbb{R}_+) \subset \mathbb{R}^J$, it is open in $\mathbb{R}^J$ with the product top. Thus $\pi_\gamma^{-1}((0,+\infty)) \cap Z$ is open in Z for the subspace top. $Z \subset \mathbb{R}^J$.

claim: ① $\pi_\gamma^{-1}((0,\infty)) \cap Z$ is an open nbd of z_0

② $F^{-1}(\pi_\gamma^{-1}((0,\infty)) \cap Z) \subset U$

① notice that $x_0 := F^{-1}(z_0)$ $\therefore F(x_0) = z_0$.

Then, $\pi_\gamma(z_0) = \pi_\gamma(F(x_0)) = f_\gamma(x_0) > 0$ $\therefore z_0 \in \pi_\gamma^{-1}((0,\infty))$

Thus $z_0 \in \pi_\gamma^{-1}((0,\infty)) \cap Z$

② if $w_0 \in \pi_\gamma^{-1}((0,\infty)) \cap Z^{=F(X)}$, then $\exists y_0 \in X$ s.t. $F(y_0) = w_0$

moreover, $w_0 \in \pi_\gamma^{-1}((0,\infty)) \Rightarrow \pi_\gamma(w_0) > 0$

$$\left.\begin{array}{l} \pi_\gamma(w_0) > 0 \\ \quad \| \\ \pi_\gamma(F(y_0)) = f_\gamma(y_0) \end{array}\right\} f_\gamma(y_0) > 0$$

$\Rightarrow y_0 \in U$, b/c $f_\gamma|_{X \setminus U} = 0$ $\Rightarrow w_0 = F(y_0) \in F(U)$

Thus $\pi_\gamma^{-1}((0,\infty)) \cap Z \subset F(U)$

Now since z_0 is any pt in Z, U is any nbd of $F^{-1}(z_0)$, we conclude that $F^{-1} : Z \to X$ is cont also. #

Recall in § 20, we showed that $\mathbb{R}^\omega$ with product top is metrizable, but we didn't know what kind of top is for sure to be metrizable. The Urysohn metrization theorem gives us some sufficient conditions :

Thm 34.1 Every regular space X with a countable basis is metrizable.

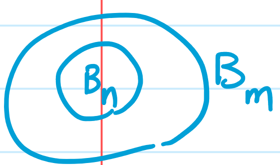
Thm 32.1, regular with countable basis is normal, thus can apply Urysohn lemma

Pf: Because of the imbedding thm, we just have to show that there is a countable collection of continuous fun $f_n : X \to [0,1]$ having the property that $\forall x_0 \in X$ and a nbd U of x_0, there exists an index n s.t. $f_n(x_0) > 0$ and $f_n|_{X \setminus U} = 0$.

This is b/c we can use such $\{f_n\}_{n \in \mathbb{Z}_+}$ to define $F: X \to \mathbb{R}^\omega := \prod_{n \in \mathbb{Z}_+} A_n$ each $A_n = \mathbb{R}$, so we want the index set to be $\mathbb{Z}_+$, i.e. countable.

Then since $\mathbb{R}^J$ with the product top is metrizable, a subspace with subspace top is again metrizable cf: § 21 ex # 1, which implies that X, being homeomorphic to $F(X)$, is metrizable.

Let $\{B_n\}$ be a countable basis for X. For each pair (n, m) of indices for which $\overline{B_n} \subset B_m$, apply the Urysohn lemma to choose a cont. fun $g_{n,m} : X \to [0,1]$ s.t. $g_{n,m}|_{\overline{B_n}} \equiv 1$ and $g_{n,m}|_{X \setminus B_m} = 0$



It is possible that the pair (n, m) doesn't have the property that $\overline{B_n} \subset B_m$, then we don't have $g_{n,m}$ for this pair. Thus the collection $\{g_{n,m}\}_{(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+}$ is indexed with a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$.

Now given any $x_0 \in X$ and a nbd U of x_0. we can choose a basis element B_m s.t. $x \in B_m$ and $B_m \subset U$. Now since X is regular, we can find a basis element B_n s.t. $x \in B_n$ and $\overline{B_n} \subset B_m$. Then for this pair (n, m), $g_{n,m}$ is defined, and from the definition, we have $g_{n,m}(x_0) > 0$ and $g_{n,m}|_{X \setminus U} = 0$ ($\because B_m \subset U$, and $g_{n,m}|_{X \setminus B_m} = 0$) which is the property we desired.

Moreover, b/c the indices are a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$, it's countable, and we can reindex them with $\mathbb{Z}_+$ and get a countable collection of functions $\{f_n\}_{n \in \mathbb{Z}_+}$. (i.e. every f_n is some $g_{k,l}$)

#

The "iff" condition for metrizable is the Nagata-Smirnov metrization theorem in §40, which says,

A space X is metrizable iff X is regular, and has a basis that's countably locally finite.

(recall, last time)

有興趣的同學可以自己讀.

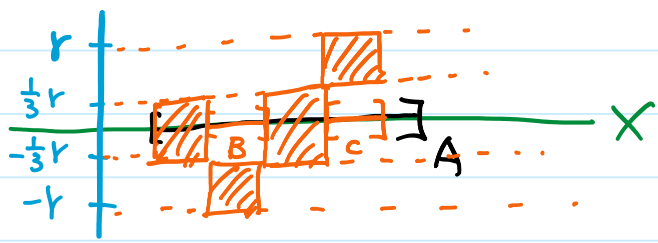
Lemma Let X be a normal space, $A \subset X$ closed subspace.

If f is a cont. fun $A \to [-r, r]$, then $\exists$ cont. $g: X \to \mathbb{R}$ s.t. $|g(x)| \le \frac{r}{3}$, $\forall x \in X$, $|g(a) - f(a)| \le \frac{2}{3}r$, $\forall a \in A$.

Pf: Given $f: A \to [-r, r]$, consider the subinterval $[-r, -\frac{r}{3}]$, $[-\frac{r}{3}, \frac{r}{3}]$ and $[\frac{r}{3}, r]$. Define $B \overset{cA}{=} f^{-1}([-r, -\frac{r}{3}])$ and $C \overset{cA}{=} f^{-1}([\frac{r}{3}, r])$. Because f is cont, B and C are closed in A. Since A is closed in $X \Rightarrow B$ & C closed in X.

Since X is normal, Urysohn Lemma says $\exists$ a continuous fun $g: X \to [-\frac{r}{3}, \frac{r}{3}]$ such that $g|_B \equiv -\frac{r}{3}$, $g|_C \equiv \frac{r}{3}$.

So $|g(x)| \le \frac{r}{3}$, $\forall x \in X$ as desired. Moreover, it also satisfies $|g(a) - f(a)| \le \frac{2}{3} r$, $\forall a \in A$:



① if $a \in B$, then $g(a) = -\frac{r}{3}$ and $-r \le f(a) \le -\frac{r}{3}$
$\Rightarrow |f(a) - g(a)| \le \frac{2}{3}r$

② if $a \in C$, then $g(a) = \frac{r}{3}$, and $\frac{r}{3} \le f(a) \le r$
$\Rightarrow |f(a) - g(a)| \le \frac{2}{3}r$

③ if $a \in A \setminus B \setminus C$, then $-\frac{r}{3} \le f(a) \le \frac{r}{3}$, $-\frac{r}{3} \le g(a) \le \frac{r}{3} \Rightarrow |f(a) - g(a)| \le \frac{2}{3}r$ ※

Thm 35.1 (A) (Tietze extension theorem)

X: normal, $A \subset X$ closed. then any continuous fun $f: A \to [a,b]$ may be extended a cont fun: $X \to [a,b]$.

pf: W.L.O.G. we can replace $[a,b]$ by $[-1,1]$.

Let $f: A \to [-1,1]$ be a cont. fun. then by the previous lemma, $\exists \overset{\text{cont.}}{g_1}: X \to [-\frac{1}{3}, \frac{1}{3}]$ s.t. $|g_1(x)| \le \frac{1}{3}$ $\forall x \in X$ and

$|f(a) - g_1(a)| \le \frac{2}{3}$, $\forall a \in A$.

Next, consider the cont. function $f - g_1: A \to [-\frac{2}{3}, \frac{2}{3}]$.

Apply Lemma again, we get a cont fun $g_2: X \to [\frac{1}{3}(-\frac{2}{3}), \frac{1}{3}(\frac{2}{3})]$

s.t. $|g_2(x)| \le \frac{1}{3} \cdot \frac{2}{3}$ $\forall x \in X$ and $|(f - g_1)(a) - g_2(a)| \le \frac{2}{3} \cdot \frac{2}{3}$, $\forall a \in A$

Repeat this process, we have cont funs $g_1(x), \cdots, g_n(x)$ defined on X, $|g_n(x)| \le (\frac{1}{3})(\frac{2}{3})^{n-1}$ $\forall x \in X$ and

$|f(a) - g_1(a) - \cdots - g_n(a)| \le (\frac{2}{3})^n$ for $a \in A$.

Applying Lemma to this fun $(f - g_1 - \cdots - g_n): A \to [-(\frac{2}{3})^n, (\frac{2}{3})^n]$ new r

we obtain a new fun $g_{n+1}(x)$, s.t. $|g_{n+1}(x)| \le \frac{1}{3} \cdot (\frac{2}{3})^n$

and $|f(a) - g_1(a) - \cdots - g_n(a) - g_{n+1}(a)| \le \frac{2}{3} \cdot (\frac{2}{3})^n$ $\forall a \in A$.

By induction, the fun $g_n(x)$ is defined $\forall n$, on X.

and $|g_n(x)| \le \frac{1}{3} \cdot (\frac{2}{3})^{n-1}$ $\forall x \in X$ with $|f(a) - g_1(a) - \cdots - g_n(a)| \le (\frac{2}{3})^n$

Now consider their sum $g(x) := \sum_{n=1}^{\infty} g_n(x)$. we claim that this is the extension we want : need to show

1° $g(x)$ is well-defined, i.e. $\sum_{n=1}^{\infty} g_n(x)$ conv. $\forall x \in X$

2° $g(x)$ is cont on X

3° $g(a) = f(a)$ $\forall a \in A$

1° for any $x \in X$, $\left| \sum_{n=1}^{\infty} g_n(x) \right| \leq \sum_{n=1}^{\infty} |g_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$ geometric series

2° Thm 21.6 Let $f_n : X \to Y$ be a sequence of cont. fns from X to a metric space Y. If $\{f_n\}$ conv. uniformly to f, then f is continuous.

So we will use the partial sum as the sequence: $S_k(x) := \sum_{n=1}^{k} g_n(x)$

and $g(x) := \sum_{n=1}^{\infty} g_n(x) = \lim_{k\to\infty} S_k(x)$.

To show that $\{S_k\}$ converges uniformly :

for $k > n$, $|S_k(x) - S_n(x)| = \left| \sum_{i=n+1}^{k} g_i(x) \right| \leq \sum_{i=n+1}^{k} |g_i(x)|$

$$\leq \sum_{i=n+1}^{k} \frac{1}{3}\left(\frac{2}{3}\right)^{i-1} < \underbrace{\sum_{i=n+1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{i-1}}_{= \frac{\frac{1}{3}\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = \left(\frac{2}{3}\right)^n}$$

$$\Rightarrow \left| \overbrace{\lim_{k\to\infty} S_k(x)}^{g(x)} - S_n(x) \right| \leq \left(\frac{2}{3}\right)^n$$

i.e. $\{S_n(x)\}$ converges to $g(x)$ uniformly, thus g is cont.

3° $|f(a) - S_n(a)| = \left| f(a) - \sum_{i=1}^{n} g_i(a) \right| \leq \left(\frac{2}{3}\right)^n$ (by construction)

$\Rightarrow \left| f(a) - \lim_{n\to\infty} S_n(a) \right| \leq \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0 \Rightarrow f(a) = g(a)$ $\forall a \in A$ #

Thm 35.1 (B) (Tietze extension theorem)

X: normal, $A \subset X$ closed. then any continuous fun $f: A \to \mathbb{R}$ may be extended a continuous fun: $X \to \mathbb{R}$

(difference being "open" vs. "closed")

pf: Since $\mathbb{R}$ is homeomorphic to $(-1,1)$, we can just prove the statement for $(-1,1)$.

i.e. $f: A \to (-1,1)$, we want to extend it to $g: X \to (-1,1)$.

Now $f: A \to (-1,1)$ can be viewed as $f: A \to [-1,1]$, and from Thm 35.1 (A), we know there is an extension, cont, $g: X \to [-1,1]$. What we need to do now is to modify g so that it becomes $X \to (-1,1)$.

Given $f: A \to (-1,1)$, by Thm 35.1(A), we have an extension, $g: X \to [-1,1]$, which is also continuous.

Consider the subset in X: $D := g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$

(i.e. the pts in X s.t. its value under g is precisely ± 1)

D is closed b/c g is cont. Moreover, $D \cap A = \emptyset$ b/c $g(a) = f(a)$ $\forall a \in A$ and $f(A) \subset (-1,1)$.

So we have two disjoint closed sets: A and D, so by Urysohn lemma (X: normal), there is a continuous fun $\psi: X \to [0,1]$ s.t. $\psi|_D \equiv 0$, $\psi|_A \equiv 1$. (ψ could be 1 at pts in $X \setminus A \setminus D$, but we don't care)

Now consider the new fun $h(x) := \varphi(x) g(x)$.
Then $h(x)$ is cont, b/c φ and g are both cont. Moreover,
we have $h(a) = \varphi(a) g(a) = 1 \cdot g(a) = f(a)$, $\forall a \in A$
so $h(x)$ is an extension of $f(x)$.

what's the image of $h(x)$?
if $x \in D$, then $h(x) = \varphi(x) g(x) = 0 \cdot g(x) = 0$
if $x \notin D$, then $|g(x)| < 1$ ($\because D := g^{-1}(-1) \sqcup g^{-1}(1)$, $|g(x)| \leq 1$)
while $0 \leq \varphi(x) \leq 1 \quad \forall x \notin D$
$\Rightarrow |h(x)| = |\varphi(x) g(x)| \leq |1 \cdot g(x)| < 1 \Rightarrow -1 < h(x) < 1$
i.e. $h : X \longrightarrow (-1, 1)$ and $h(a) = f(a) \ \forall a \in A$ ※